Building Blocks of Geometric Group Theory

Eleanor Rhoads

GGT#1: Group Presentations

DEFINITION

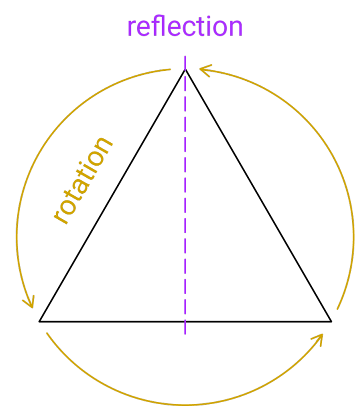
A group is a set G with an (group) operation a nice way to combine elements $*: G \times G \to G$, written as $(g, h) \mapsto g * h$ "maps to", where:

- Identity: there exists $e \in G$ the identity such that for all $g \in G$, $e * g = g = g * e$.
- Inverses: for each $g \in G$, there is a $h \in G$ inverse of g, denoted g^{-1} such that $g * h = e = h * g$.
- Associativity: for all $g, h, k \in G$, $(g * h) * k = g * (h * k)$.

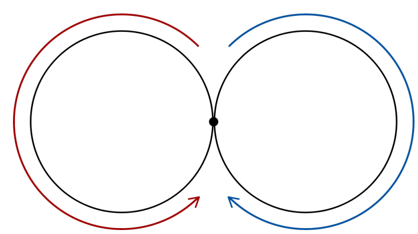
Examples—

(1) $\mathbb{R}$ with $+$, $\mathbb{R}$ excluding zero with $\times$.

(2) Symmetries maps back to itself while preserving the structure, or "movements" of a equilateral triangle.



(3) Loops starting and ending at some basepoint in a space. Can do any combination of these.



Groups aren't enough structure for $\mathbb{R}$, finite groups are covered in an abstract algebra class. In the last example groups are fitting, but the group is infinite. What do we do?

$$\text{Geometric Group Theory or GGT} = \begin{cases} \text{Viewing groups as geometric spaces our focus.} \\ \text{Studying how groups and geometric spaces interact.} \end{cases}$$

Today's Goal— Construct groups from scratch via presentations, which will give us a notion of geometry later on.

Like cooking, we need ingredients and to know how to combine them.

- $S =$ a set "alphabet" of "letters".
- A finite sequence in S is a word over S written as concatenation.
- $e =$ the empty word.
- $|w| =$ length of w.
- $S^* = \{\text{words over } S\}$.
- Operation $S^* \times S^* \to S^*$ via concatenation.

Example—

- $S = \{a, b\}$.
- $aaabba$ or a^3b^2a is a word over S.
- $|a^3b^2a| = 6$.
- ab^2 is a subword of a^3b^2a.
- $S^* = \{e, a, b, aa, ab, ba, bb, aaa, \ldots\}$.
- $(a^2, ab^2a) \mapsto a^3b^2a$.

We have an operation with a (hopeful) identity, and associativity... what about inverses?

- $S^{-1} = \{s^{-1} \text{ formal inverse of } s : s \in S\}$.
- $S^{\pm *} = (S \cup S^{-1})^*$.
- $w^{-1} =$ reverse and invert w. Socks and shoes.

- $S^{-1} = \{a^{-1}, b^{-1}\}$.
- $S^{\pm *} = S^* \cup \{a^{-1}, b^{-1}, aa^{-1}, a^{-1}a, ba^{-1}, \ldots\}$.
- $(aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1}$.

Idea— Specify words that tell us what is true in our group.

For $R \subseteq S^{\pm *}$, say $w, v \in S^{\pm *}$ are equivalent over R, $w \equiv_R v$, if we can get from w to v by adding/deleting subwords ss^{-1}, $s^{-1}s$, r, or r^{-1} for $s \in S$ or $r \in R$.

Example— $R = \{aba^{-1}b^{-1}\}$, then $ab \equiv_R aba^{-1}a \equiv_R aba^{-1}b^{-1}ba \equiv_R ba$.

DEFINITION

$S^{\pm *}$ up to equivalence over R with concatenation is the group given by the presentation $\langle S \mid R \rangle$.

This does actually describe all groups: exercise.

- Elements of $\langle S \mid R \rangle$ are equivalence classes of words.
- $S^{\pm *} \to \langle S \mid R \rangle$ via $w \mapsto \overline{w}$ is evaluation, and w represents $\overline{w}$. Will abuse notation.
- Elements of S are generators, elements of R are relators. Can also write as $w = v$, relations.

- $\langle S \mid R \rangle$ is finitely generated when S is finite our main focus, and finitely presented when S and R are finite.
- $w \in S^{\pm *}$ is trivial in $\langle S \mid R \rangle$ if $w \equiv_R e \iff \overline{w} = \overline{e}$ our identity element.

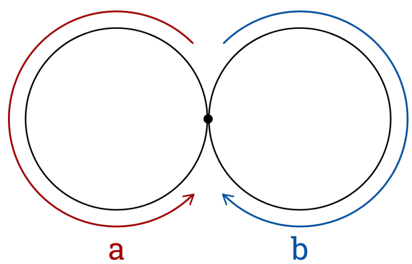
Examples—

(1) $\langle S \mid \emptyset \rangle$ or $\langle S \rangle$ is the free group on S, $F(S)$. $|S| =$ rank of $F(S)$.

Elements of $F(S) \longleftrightarrow$ reduced words, i.e. contain no ss^{-1} or $s^{-1}s$ subwords.

$\langle a \rangle$ is made precise later $\mathbb{Z}$ with $+$.

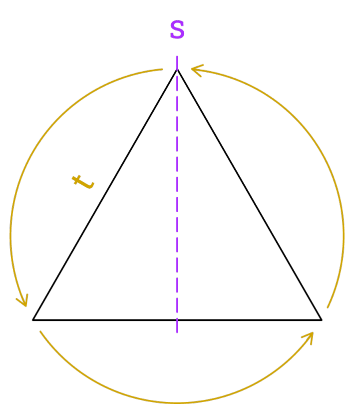
$\langle a, b \rangle$ is...



(2) $\langle a \mid a^n \rangle$ is $\mathbb{Z}$ modulo n.

(3) $\langle a, b \mid aba^{-1}b^{-1} \rangle$ or $ab = ba$ is $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. Elements $\longleftrightarrow$ words $a^k b^l$.

(4) $\langle s, t \mid s^2, t^3, stst \rangle$ is the symmetries of a triangle.



(5) $\langle a, b \mid a^2 = b \rangle$ is $\mathbb{Z}$. Uh oh!

What does it mean for two groups G and H to be “the same”?

DEFINITION

A (group) homomorphism $\varphi : G \to H$ is a map where for all $x, y \in G$, $\varphi(x *_G y) = \varphi(x) *_H \varphi(y)$. The operations are "the same".

φ is an isomorphism if it is also bijective. The sets are "the same". G and H are isomorphic "the same group", $G \cong H$, if there exists an isomorphism between them.

Example— All free groups of rank n are isomorphic, so F_n = "the" free group of rank n.

GGT#2: Cayley Graphs

Common viewpoint: groups $\longleftrightarrow$ symmetries of some object. Such as the symmetries of a triangle.

But if we are given a group, what's the object?

Today's Goal— Draw meaningful pictures for groups.

DEFINITION

A directed graph Γ is a pair (V, E) where

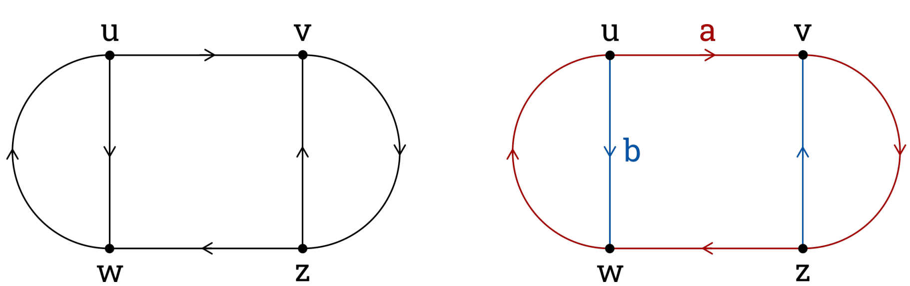
- $V =$ a set, elements are vertices.
- $E =$ ordered pairs (v_1, v_2) where $v_1, v_2 \in V$, elements are edges.

If we assign a label to each edge, then Γ is a directed labeled graph.

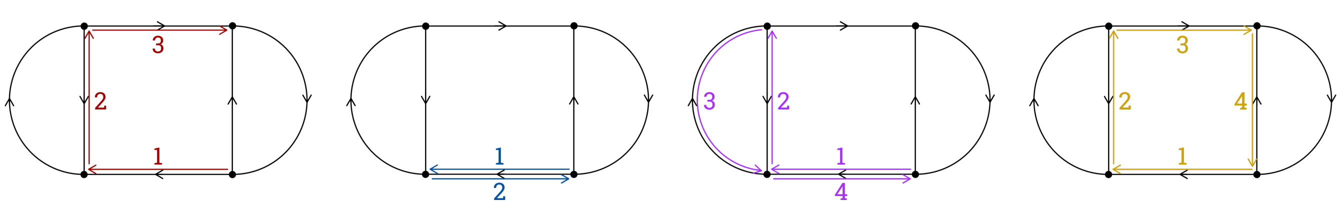
Not necessarily finite, allowing for loops and multiple edges to the same vertex.

Graphs are technically defined via sets, but it's easier to represent them with pictures.

Examples—



- A path is a sequence of adjacent edges, can include reversed orientation.
- A path is reduced if it doesn't backtrack, i.e. an edge is never directly followed by the edge backwards.
- A cycle (or loop) is a nontrivial reduced path that starts and ends at the same vertex.



DEFINITION

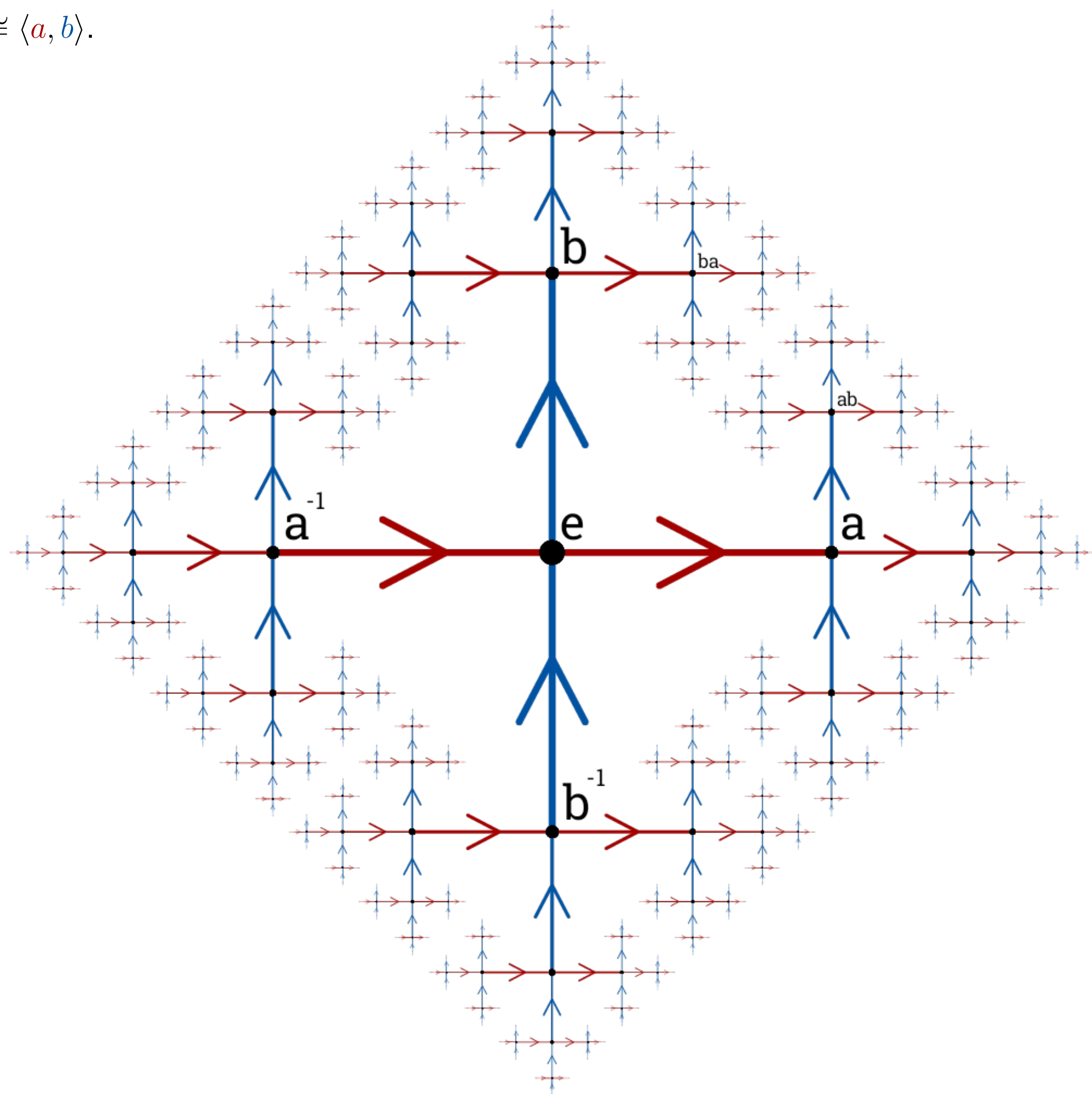
$G = \langle S \mid R \rangle$. The the Cayley graph of G with respect to S is the directed labeled graph $\Gamma(G, S)$ or $\Gamma(G)$ or Γ where

- $V = G$,
- $E = \{(g, gs)$ labeled by $s : g \in G, s \in S\}$ reverse orientation "labeled" by s^{-1}.

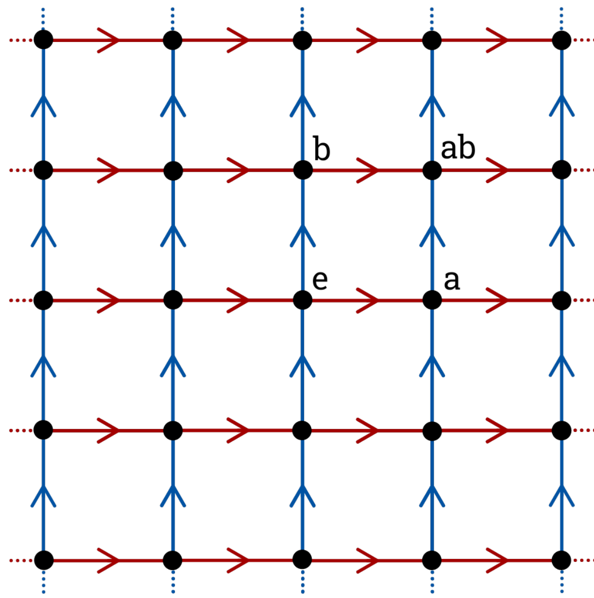
- Reduced paths $\longleftrightarrow$ reduced words, by reading labels of edges. Lengths correspond.
- Cycles $\longleftrightarrow$ reduced trivial words including each $r \in R$.
- Paths from e to g $\longleftrightarrow$ words that evaluate to g.

Examples—

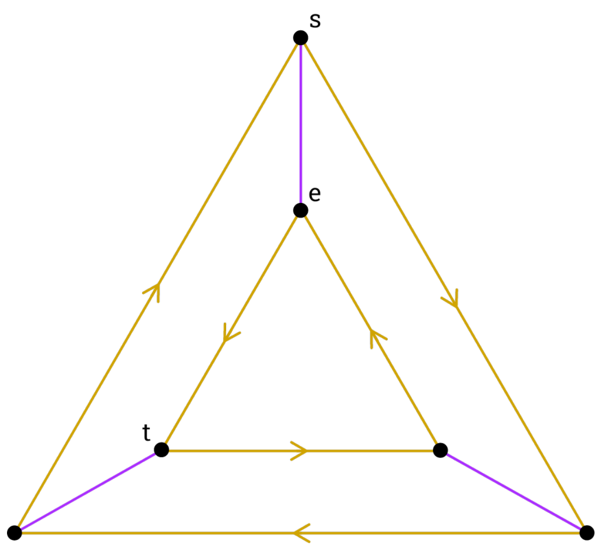
(1) $F_2 \cong \langle a, b \rangle$.



(2) $\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$. Students should try this one.



(3) Symmetries of a triangle $\cong \langle s, t \mid s^2, t^3, st = t^2 s \rangle$.



Properties—

- Γ is connected if for all $v_1, v_2 \in V$, there is a path from v_1 to v_2.
- Number of edges containing v = degree of v (loops count twice).
- Γ is n-regular if all vertices have degree n finite.
- Γ is a tree if it is connected and has no cycles.

PROPOSITION

$G = \langle S \mid R \rangle$.

(1) $\Gamma(G, S)$ is connected.

(2) If G is finitely generated, then $\Gamma(G, S)$ is $2|S|$-regular.

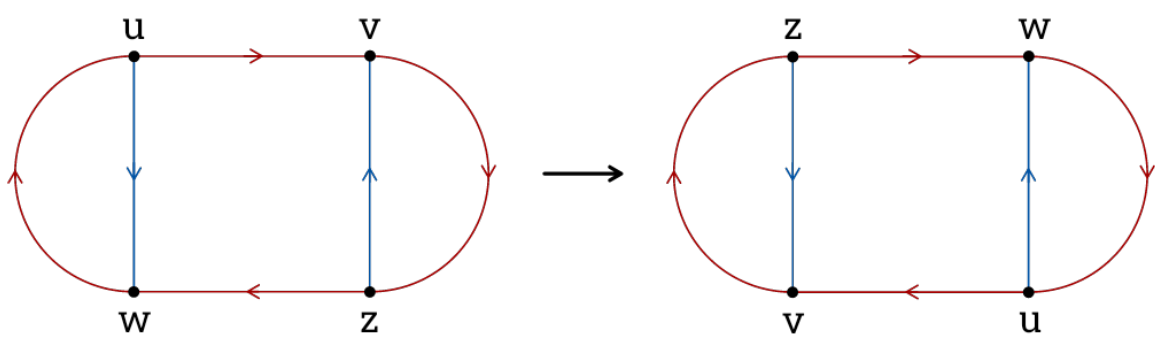
(3) $\Gamma(G, S)$ is a tree if and only if $R = \emptyset$.

In what sense does $\Gamma(G, S)$ "represent" G?

DEFINITION

An automorphism symmetry of a directed labeled graph $\Gamma = (V, E)$ is a bijective map $f : V \to V$ such that for each $(u, v) \in E$ with label l, there exists an edge $\big(f(u), f(v)\big)$ with label l.

Example— $u \mapsto z$, $v \mapsto w$, $w \mapsto v$, $z \mapsto u$.



DEFINITION

Automorphisms of Γ with composition forms a group, called the automorphism group of Γ.

THEOREM

$G = \langle S \mid R \rangle \cong \mathrm{Aut}\,\big(\Gamma(G, S)\big)$, i.e. the group of symmetries of Γ is G.

Proof— For each $g \in G$, let $\alpha_g \in \mathrm{Aut}\,\big(\Gamma(G, S)\big)$ be $h \mapsto gh$. Then $\Phi : G \mapsto \mathrm{Aut}\,\big(\Gamma(G, S)\big)$ via $g \mapsto \alpha_g$ is an isomorphism. This is a group action. $\square$

GGT#3: Quasi-Isometries

Today's Goal— Capture geometric information from G and Γ.

DEFINITION

A metric space is a set X with a metric $d : X \times X \to \mathbb{R}$ $d(x, y) =$ distance between x and y where for all $x, y, z \in X$ "points":

- $d(x, y) > 0$ if $x \neq y$ and $d(x, x) = 0$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$ triangle inequality.

Example— $\mathbb{R}$ and $\mathbb{Z}$ are metric spaces with $d(x, y) = |y - x|$.

DEFINITION

The word metric on the group $G = \langle S \mid R \rangle$ with respect to S is $d : G \times G \to \mathbb{R}$ or $\mathbb{Z}$ via

$$d(g, h) = \min\{|w| : w \in S^{\pm *} \text{ where } \overline{w} = g^{-1}h\}.$$

DEFINITION

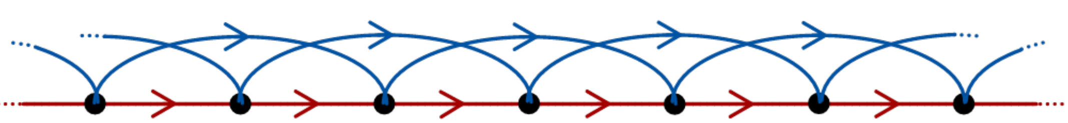
The path metric on the connected graph $\Gamma = (V, E)$ really, on V is $d : V \times V \to \mathbb{R}$ or $\mathbb{Z}$ via

$$d(u, v) = \min\{\text{lengths of paths from } u \text{ to } v\}.$$

THEOREM

The word metric on $G = \langle S \mid R \rangle$ with respect to S is the same map as the path metric on $\Gamma(G, S)$.

Example— $\mathbb{Z} \cong \langle a \rangle \cong \langle a, b \mid a^2 = b \rangle$.



With either view, G is a metric space i.e. a geometric object. But to what degree does this depend on S?

What does it mean two metric spaces (X, d_X) and (Y, d_Y) to be "the same"?

DEFINITION

An isometry is a bijective injectivity implied map $f : X \to Y$ where for all $p, q \in X$,

$$d_X(p,q) = d_Y\big(f(p), f(q)\big).$$

The map preserves distances. X and Y are isometric if there exists an isometry between them.

Problem— This is too strong of a notion of "sameness" for groups as metric spaces!

DEFINITION

$K \geq 1$, $L, D \geq 0$. A (K, L)-quasi-isometry is a map $f : X \to Y$ where

- for all $p, q \in X$, the map coarsely preserves distances

$$\frac{1}{K} \cdot d_X(p,q) - L \leq d_Y\big(f(p), f(q)\big) \leq K \cdot d_X(p,q) + L.$$

- for each $y \in Y$, there exists $x \in X$ such that the map is coarsely surjective

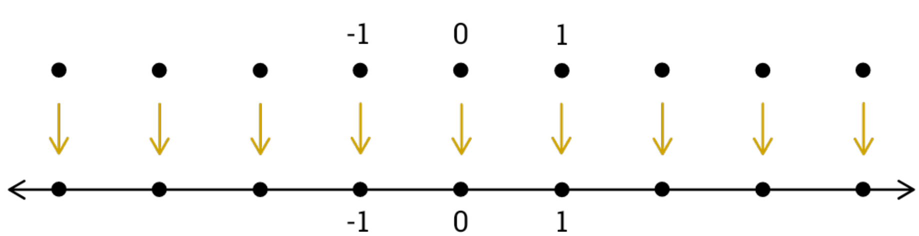
$$d_Y\big(y, f(x)\big) \leq D.$$

Does direction matter? No. Quasi-isometry is an equivalence relation.

DEFINITION

X and Y are quasi-isometric if there exists a quasi-isometry between them. We don't really care about the constants.

Example— $\mathbb{Z} \to \mathbb{R}$ via $n \mapsto n$ is a $(1, 0)$-quasi-isometry. What's D? $\frac{1}{2}$.



THEOREM

If $\langle S \mid R\rangle \cong \langle T \mid P\rangle$ and S and T are finite, then $\langle S \mid R\rangle$ with respect to S is quasi-isometric to $\langle T \mid P\rangle$ with respect to T.

- Two groups being quasi-isometric does not mean they are isomorphic!
- Can also consider whether a group is quasi-isometric to other metric spaces.

What if we want to consider the "whole" graph?

DEFINITION

the geometric realization of a connected graph $\Gamma = (V, E)$ is Γ as a metric space, with the metric $d : \Gamma \times \Gamma \to \mathbb{R}$ which extends the path metric to edges by viewing them as copies of $[0, 1]$.

Example—

THEOREM

If $G = \langle S \mid R\rangle$ is finitely generated, then G is quasi-isometric to the geometric realization of $\Gamma(G, S)$.

So from a coarse geometric perspective, G and Γ are the same thing.

One Application— The word problem.

$G = \langle S \mid R\rangle$, finitely generated. Given $u, v \in S^{\pm *}$ can you algorithmically determine if u and v represent the same element equivalence class in G? For example, reduce words for F_n.

$\overline{u} = \overline{v} \iff \overline{uv^{-1}} = \overline{e}$, so we can rephrase as... given $w \in S^{\pm *}$ can you determine if w is trivial in G?

In general, no! (Novikov, 1955)

THEOREM

Suppose G and H are quasi-isometric finitely generated groups.

(1) If G is finitely presented, so is H.

(2) If G has a solvable word problem, so does H. With the same "complexity".

Recommended Reading— *Office Hours with a Geometric Group Theorist*, 2017, Clay & Margalit.

GGT#1: Group Presentations — Exercises

(1) (a) For $S = \{s, t\}$ and $R = \{s^2, t^3, stst\}$, show that $st \equiv_R t^2 s$.

(b) For $S = \{a, b, c\}$ and $R = \{cac^{-1}b^{-1}, cbc^{-1}a^{-1}\}$, show that $abc \equiv_R cba$.

(2) Show that every word over $\{s, t\}$ is equivalent over $\{s^2, t^3, stst\}$ to one of the following:

$$\{e, t, t^2, s, st, st^2\}.$$

(In particular, this shows that $\langle s, t \mid s^2, t^3, stst \rangle$ is a finite group.)

(3) (a) Show that the operation on $\langle S \mid R \rangle$ is well-defined. That is, for $u_1, u_2, v_1, v_2 \in S^{\pm *}$, show that if $\overline{u_1} = \overline{u_2}$ and $\overline{v_1} = \overline{v_2}$ then $\overline{u_1 v_1} = \overline{u_2 v_2}$.

(b) Show that $\langle S \mid R \rangle$ is indeed a group.

(4) For the two following groups, determine whether the group is finite. If it is finite, give a word that represents each group element. If it is not finite, give an infinite list of words that each represent a different group element. (You do not need proof.)

(a) $\langle a, b \mid a^2, b^2 \rangle$.

(b) $\langle a, b \mid a^2, b^2, aba^{-1}b^{-1} \rangle$.

(5) Let $G = \langle S \mid R \rangle$ and H be groups. Suppose $\varphi : G \to H$ and $\psi : G \to H$ are homomorphisms such that $\varphi(\overline{s}) = \psi(\overline{s})$ for each $s \in S$. Show that φ and ψ must be the same map, i.e. $\varphi(g) = \psi(g)$ for all $g \in G$. (This shows that homomorphisms are determined by where they send generators.)

(6) Show that $\varphi : \mathbb{Z} \to \langle a \rangle$ via $n \mapsto \overline{a^n}$ is an isomorphism. (*Note*: $a^0 = e$.)

(7) Show that every group admits a presentation. In other words, show that every group is isomorphic to a group given by a presentation. (The presentation does not need to efficient!)

(8) (a) Describe what trivial words in $\langle a, b \mid aba^{-1}b^{-1} \rangle$ look like.

(b) Show that any homomorphism $\varphi : \langle x, y \rangle \to \langle a, b \mid aba^{-1}b^{-1} \rangle$ is not injective.

(9) **If you have taken abstract algebra:** Recall that a subgroup $N \leq G$ is normal if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$, and we can then define the quotient group G/N.

(a) Let $R \subseteq F(S)$. The normal closure of R, denoted $\langle\langle R \rangle\rangle$, is the smallest normal subgroup that contains R. Describe what elements of $\langle\langle R \rangle\rangle$ look like.

(b) Show that $\langle S \mid R \rangle \cong F(S)/\langle\langle R \rangle\rangle$.

(This can be taken as an alternate definition for a group presentation. Note that when combined with Problem 5, this shows that every group is the quotient of a free group.)

GGT#2: Cayley Graphs — Exercises

(1) (a) Draw the Cayley graph of $\langle s, t \mid s^2, t^4, st = t^{-1}s \rangle$.

(b) Use the Cayley graph to find a reduced word of length ≥ 10 which is trivial in the group.

(2) Draw the Cayley graph of $\langle a, b \mid a^2, b^2 \rangle$.

(3) Draw the Cayley graph of $\langle a, b, c \rangle$.

(4) Draw the Cayley graph of $\langle a, b, c \mid ab = ba \rangle$. (This is hard! This is sometimes called the "tree of flats".)

(5) Let $F_2 = \langle a, b \rangle$ and $\Gamma = \Gamma(F_2, \{a, b\})$. For each $g \in F_2$, we have an automorphism $\alpha_g \in \mathrm{Aut}(\Gamma)$ via $h \mapsto gh$. Also recall that each element of F_2 corresponds to a reduced word over $\{a, b\}$.

(a) Consider $a \in F_2$. Describe what α_a does to Γ.

(b) Consider the map $\beta : F_2 \to F_2$ which swaps a and b as they appear in a reduced word. For example, $\beta(aba^{-1}b^{-1}) = bab^{-1}a^{-1}$. Is β an automorphism of Γ? Why or why not?

(6) Let $\Gamma = (V, E)$ be a directed labeled graph. Show that $\mathrm{Aut}(\Gamma)$ is indeed a group.

GGT#3: Quasi-Isometries — Exercises

(1) Let $G = \langle S \mid R \rangle$. Show that G with the word metric (with respect to S) is indeed a metric space.

(2) For a metric space X with metric d, $r > 0$, and $x \in X$, the sphere of radius r centered at x is

$$S_r(x) = \{y \in X : d(x, y) = r\}.$$

(a) For $\langle a, b \rangle$ and n a positive integer, describe $S_n(e)$ as a set. What does $S_n(e)$ look like in the Cayley graph? How many elements are in $S_n(e)$?

(b) For $\langle a, b \mid aba^{-1}b^{-1} \rangle$, describe $S_n(e)$ in the same manner as above.

(3) Find a quasi-isometry $f : \mathbb{R} \to \mathbb{Z}$. What are your constants?

(4) Let $f : \langle a \rangle \to \langle x, y \mid x^2 = y \rangle$ be the map $a \mapsto x$. Show that f is a quasi-isometry.

(5) Let $\Gamma = (V, E)$ be connected. Show that the geometric realization of Γ is quasi-isometric to Γ with the path metric.

(6) Let $G = \langle S \mid R \rangle$ and $\Gamma = \Gamma(G, S)$. Recall that for the proof of $G \cong \mathrm{Aut}(\Gamma)$, we defined an automorphism for each $g \in G$ called $\alpha_g \in \mathrm{Aut}(\Gamma)$ where $h \mapsto gh$. Show that $\alpha_g : G \to G$ is an isometry.