Constructing the Surreal Numbers

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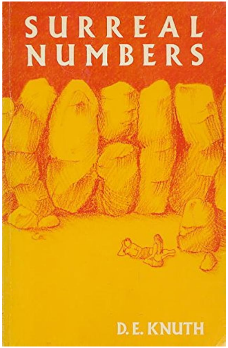
Introduction

The Main Characters

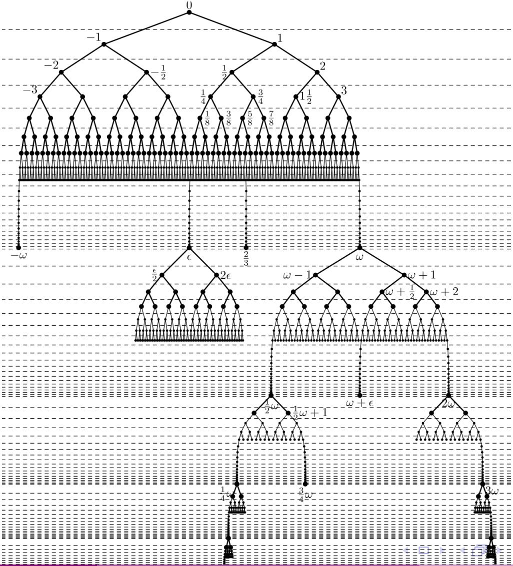
- John Conway
- D.E. Knuth
- Alice and Bill



Construction Goals

1. All the Real Numbers (and then some)
    - Infinitesimal and Ordinals
2. Ordering
    - For any a, b then either $b \leq a$, or $a \leq b$
3. Two Binary Operations: Addition and Multiplication
    - $a + b = b + a$ and $a \cdot b = b \cdot a$
    - $a(b + c) = ab + ac$
    - If $a \neq 0$ then there exists $-a$ and a^{-1} such that $a + (-a) = 0$ and $aa^{-1} = 1$

Our Road-map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$\frac{5}{8}$
$\frac{7}{8}$
$-\omega$
ϵ
$\frac{2}{3}$
ω
$\frac{\epsilon}{2}$
2ϵ
$\omega - 1$
$\omega + 1$
$\omega + \frac{1}{2}$
$\omega + 2$
$\frac{1}{2}\omega$
$\frac{1}{2}\omega + 1$
$\omega + \epsilon$
2ω
$\frac{1}{4}\omega$
$\frac{3}{4}\omega$
3ω

Constructing the Surreals

The First Definition

Definition (Surreal Number)

A surreal number x is a tuple of sets $x = (X_L | X_R)$ such that

1. Each $x_L \in X_L$ and $x_R \in X_R$ is a previously created surreal number
2. For any $x_L \in X_L$ and $x_R \in X_R$ $x_L \ngeq x_R$

The Ordering

Definition

Given two surreal numbers $x = (X_L|X_R)$ and $y = (Y_L|Y_R)$ we say $x \leq y$ if for any $x_L \in X_L$ $x_L \not\geq y$ and for any $y_R \in Y_R$ $x \not\geq y_R$.

Day 0

We don't have any surreal numbers yet! So how do we use our definition?

Definition (Empty Set)

The *empty set* is the unique set with no elements. It is denoted by $\emptyset$.

$$(\emptyset|\emptyset) = 0$$

Day 1

$$(\{0\}|\emptyset) = 1$$

$$(\{0\}|\{0\}) \stackrel{?}{=} 0$$

$$(\emptyset|\{0\}) = -1$$

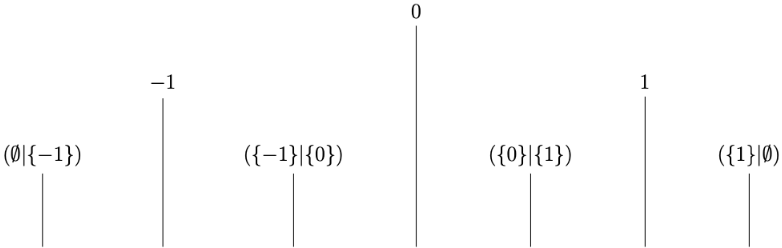
Non-Unique Representation

Definition

Given two surreal numbers $x = (X_L|X_R)$ and $y = (Y_L|Y_R)$, if $x \leq y$ and $y \leq x$ then we say $x \equiv y$.

$$(\emptyset|\emptyset) \equiv (\{0\}|\{0\}) = (\{(\emptyset|\emptyset)\}|(\{\emptyset|\emptyset)\})$$

Day 2



Addition

Definition (Addition)

$$x + y = \Big((X_L + y) \cup (Y_L + x) \Big| (X_R + y) \cup (Y_R + x)\Big)$$

$$-x = (-X_R | -X_L)$$

Additive Identity

Let $x = (\emptyset|\emptyset)$

$$\begin{aligned} x + x &= \Big((\emptyset + \{(\emptyset|\emptyset)\}) \Big| (\emptyset + \{(\emptyset|\emptyset)\}) \Big) \\ &= \Big(\{(\emptyset|\emptyset)\} \Big| \{(\emptyset|\emptyset)\} \Big) \\ &\equiv (\emptyset|\emptyset) \end{aligned}$$

Multiplication

Definition (Multiplication)

Let $x = (X_L|X_R)$ and $y = (Y_L|Y_R)$ be surreal numbers. Then $x \cdot y = (XY_L|XY_R)$ is given by

$$XY_L = \{X_L \cdot y + x \cdot Y_L - X_L \cdot Y_L\} \cup \{X_R \cdot y + x \cdot Y_R - X_R \cdot Y_R\}$$

$$XY_R = \{X_L \cdot y + x \cdot Y_R - X_L \cdot Y_R\} \cup \{X_R \cdot y + x \cdot Y_L - X_R \cdot Y_L\}$$

Multiplication Cont.

Why does this definition make sense?
Let $x_L \in X_L$ and $y_L \in Y_L$. We want to show $x_L \cdot y + xy_L - x_L \cdot y_L \in XY_L$.

$$\begin{aligned} & x_L \cdot y + xy_L - x_L \cdot y_L \leq xy \\ \iff & x_L \cdot y + xy_L - x_L \cdot y_L - xy \leq 0 \\ \iff & x_L(y - y_L) - x(y - y_L) \leq 0 \\ \iff & (x_L - x)(y - y_L) \leq 0 \end{aligned}$$

Day n

What numbers do we create on the n^{th} day?

1. $\pm n$
2. $\pm\dfrac{1}{2^{n-1}}$
3. In general, if x, y were created before day n, we make $\dfrac{x+y}{2}$

Problem! On any finite day we can only create numbers of the form $\dfrac{a}{2^k}$ where $a \in \mathbb{Z}$ and $k \leq n$.

The Birthday Problem

Remark (The Birthday Problem)

Let's consider the following collection of surreal numbers

$$(\{-1\}|\{1\}), \quad (\{-1\}|\{2,5,7\}), \quad (\{-100\}|\{10,000\})$$

In general, given $x = (\{x_L\}|\{x_R\})$ *we define* x *to be the earliest created surreal number such that* $x_L \leq x \leq x_R$.

Infinite Days

Day ω

After infinitely many days, I can now create infinite sets!

$$\frac{1}{3} = \left(\left\{\frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \dots\right\} \middle| \left\{\frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \dots\right\}\right)$$

$$\pi = \left(\left\{3\frac{1}{8}, 3\frac{9}{64}, 3\frac{289}{2048}, \dots\right\} \middle| \left\{3\frac{1}{2}, 3\frac{1}{4}, 3\frac{3}{16}, 3\frac{5}{32}, \dots\right\}\right)$$

Infinities and Infinitesimals

$$\omega = (\{1, 2, 3, \ldots\} | \emptyset)$$

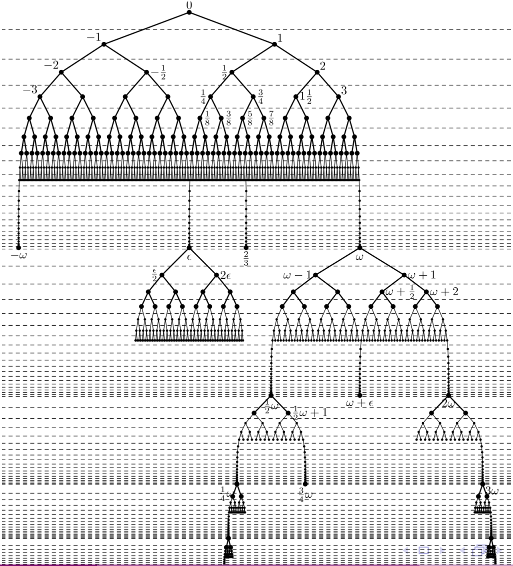
$$\epsilon = \left(\{0\} \middle| \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}\right)$$

$$\epsilon + 1 = \left(\{1\} \middle| \left\{\frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \ldots\right\}\right)$$

$$\omega + 1 = (\{\omega\} | \emptyset)$$

Conclusions

Our Road-map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$\frac{1}{2}\omega$
$\frac{1}{2}\omega + 1$
$\omega + \epsilon$
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$\frac{3}{4}\omega$
3ω

Infinity and Beyond

$$2\omega = (\{\omega, \omega + 1, \omega + 2, \omega + 3, \ldots\} | \emptyset)$$

$$\omega^2 = (\{\omega, 2\omega, 3\omega, 4\omega, \ldots\} | \emptyset)$$

$$\omega^\omega = (\{\omega, \omega^2, \omega^3, \omega^4, \ldots\} | \emptyset)$$

What Breaks?

- Topology
- Completeness
- Integration

Thank you!

Refrences

[1] D.E. Knuth. *Surreal Numbers: How Two Ex-students Turned on to Pure Mathematics and Found Total Happiness : a Mathematical Novelette.* Addison-Wesley Publishing Company, 1974.

[2] Simon Rubinstein-Salzedo and Ashvin Swaminathan. Anaysis on surreal numbers, 2015.

[3] Claus Tøndering. Surreal numbers - an introduction, 2019.